1. Let $(H, < .,. >)$ be a Hilbert space. Let $A \in B(H)$. Prove that:

   a) $KerA^* = (rangeA)^{\perp}$.

   b) $Ker(A^*A) = KerA$.

   c) If dim $H < \infty$, then $A, A^*A, \sqrt{A^*A}$ have the same rank, i.e., the dimensions of their ranges are the same.

   d) $(KerA)^{\perp}$ = the closure of range $A^* = \overline{rangeA^*}$.

   Solution:

   (a) Note $\xi \in KerA^* \Leftrightarrow A^*\xi = 0 \Leftrightarrow < A^*\xi, \eta >= 0, \forall \eta \in H \Leftrightarrow < \xi, A\eta >= 0, \forall \eta \in H \Leftrightarrow \xi \in (ranA)^{\perp}$.

   (b) Obviously $KerA \subseteq KerA^*A$. By 1(a), $KerA^*A = Ran(A^*A)^{\perp}$. Thus, $\xi \in KerA^*A$ implies $< \xi, A^*A\eta >= 0, \forall \eta \in H$ from which it follows that $A\xi = 0$ so that $\xi \in KerA$.

   (c) Use Rank-Nullity theorem to conclude that $dim(KerA^*A) + dim(RanA^*A) = dim(H) = dim(KerA) + dim(RanA)$ so that rank of A^*A equals rank of A. Let $T = \sqrt{A^*A}$. Again note that $KerT \subseteq KerA^*A$ obviously. Now $\xi \in KerA^*A \Rightarrow T^2\xi = 0 \Rightarrow < T^2\xi, \xi >=< T\xi, T\xi >= 0$ and hence $\xi \in KerT$.

   (d) For any $\xi \in H, \eta \in KerA, < A^*\xi, \eta >= 0$ so that $RanA^* \subseteq KerA^{\perp}$ which in turn implies $\overline{RanA^*} \subseteq KerA^{\perp}$. The reverse inclusion follows from 1(a).

   $\square$

2. Let $(H, < .,. >)$ be a Hilbert space. Let $A \in B(H)$.

   a) Show that if A is normal then $spr(A) = \|A\|$.

   b) If A is self adjoint and $A^n = 0$ for some $n \geq 1$, then $A = 0$.

   Solution:

   (a) If a is a normal element of a unital C^*-algebra A, then recall that there is a unique unital $*$-isomorphism $\phi : C(Spr(a)) \to A$ (known as functional calculus at a) such that $\phi(z) = a$ where z is the inclusion map of $Spr(a)$ in $\mathbb{C}$ and moreover, ϕ is isometric and image of ϕ is the C^*-subalgebra of A generated by 1 and a. From this we immediately conclude that if $A \in B(H)$ is normal, then $\|A\| = Spr(A)$.

(b) $\|A\| = spr(A) = lim_{m\to\infty}\|A^m\| = 0$.

□

3. Show that a Banach space X is reflexive iff its dual X^* is reflexive.

Solution: See Functional Analysis, S. Kesavan corollary 5.3.3, page-146 □

4. a) Show that every Banach space X is isometrically isomorphic to a closed linear subspace of $C(E)$ where E is a compact Hausdorff space.

   b) Let $X = C[0,1]$,(continuous real valued functions on $[0,1]$) with the uniform norm $\|x\| = sup_{0\le t\le 1}|x(t)|$. Its dual X^* may be identified as $X^* \cong NBV[0,1] := \{g : [0,1] \to \mathbb{R}, g(0) = 0, g$ of bounded variation and right continuous on $[0,1]\}$, with the total variation norm. Show that if $g \in X^*$ and $g(x) \ge 0, x \in X$, then g is given by a non-decreasing function on $[0,1]$.

   c) If $\{\mu_n : n \ge 1\}$ is a sequence of probability measures on $[0,1]$, show tht there-exists a subsequence $\{n_k\} \subset \{n\}$ and a probability measure μ on $[0,1]$ such that
$$\int_0^1 f(t)d\mu_{n_k}(t) \longrightarrow \int_0^1 f(t)d\mu(t), \forall f \in C[0,1].$$

Solution:

(a) Suppose that X is a Banach space and let $E = (X^*)_1$, the closed unit ball of X^* and by the Banach-Alaoglu theorem, E is compact in weak* topology. Define a map $\beta : X \longrightarrow C(E)$ by $\beta(a)(f) = f(a)$ for $a \in X, f \in E$. Linearity of β is obvious. Further for $a \in X, \|\beta(a)\|_\infty = Sup_{f\in E}|\beta(a)(f)| = Sup_{f\in E}|f(a)| = \|a\|$, showing that β is isometry. Hence the proof.

(b) See B.V.Limaye Functional Analysis, Theorem 14.5, page-245.

(c) For each n, define $\phi_n : C[0,1] \to \mathbb{C}$ by $\phi_n(f) = \int f d\mu_n$. Note that $|\phi_n(f)| \le \|f\|_\infty$ so that each $\phi_n \in C[0,1]^*$. By an appeal to the uniform boundedness principle we see that $\{\phi_n\}$ is a bounded sequence in $C[0,1]^*$. We recall that every bounded sequence in the dual of a seperable Banach space has a weak* convergent subsequence. Consequently, $\{\phi_n\}$ has a weak* convergent subsequence, say, $\{\phi_{n_k}\}$ and let $\phi_{n_k} \to \phi \in C[0,1]^*$ in weak* topology. Thus $\phi(f) = lim_{n\to\infty}\int f d\mu_n, \forall f \in C[0,1]$. Thus ϕ is a positive linear functional on $C[0,1]$ and Riesz Representation theorem says that there is a nice positive measure μ on $[0,1]$ such that $\phi(f) = \int f d\mu$. Note further $\mu([0,1]) = \phi(1) = lim\phi_{n_k}(1) = lim\mu_{n_k}([0,1]) \in [0,1]$ where 1 denotes the constant function 1 on $[0,1]$ so that μ is a probability measure. Thus $\int f d\mu_{n_k} \to \int f d\mu$. Hence the proof.

5. Let X be a Banach space and $A \in B(X)$. For $t \geq 0$, define $S_t := \exp(tA)$. Show that $S_t \in B(X), S_{t_1+t_2} = S_{t_1} \circ S_{t_2}$, and for all $x \in X$, $\lim_{t \to 0} S_t x = x$.

Solution:

The following Lemma will be useful.

Lemma 0.1 *Let A be a unital Banach algebra. If $a \in A$ and $f : \mathbb{R} \to A$ is differentiable, $f(0) = 1, f'(t) = af(t), f(t)$ commutes with a $\forall t \in \mathbb{R}$, then $f(t) = e^{ta}$.*

Proof: Let $g_1, g_2 : \mathbb{R} \to A$ be differentiable maps such that $g_i(0) = 1, g_i'(t) = ag_i(t), i = 1, 2$ and if $g_1(t)$ commutes with a for all t, then the map $h : \mathbb{R} \to A$ given by $h(t) = g_1(t)g_2(-t)$ is differentiable with zero derivative and so h is constant (for if $\tau \in A^*$, then the map from $\mathbb{R} \to \mathbb{C}$ given by $t \to \tau(h(t))$ is differentiable with zero derivative so that $\tau(h(t)) = \tau(h(0)), \forall t \in \mathbb{R}$ and since $\tau \in A^*$ is arbitrary, we have that h is constant). So, $h(t) = h(0) = 1$ implies $g_1(t)g_2(-t) = 1$. If we take $g_1(t) = g_2(t) = e^{ta}$, then we have $e^{ta}e^{-ta} = 1$. With $g_1 = f, g_2(t) = e^{ta}$ we see that $f(t)e^{-ta} = 1$ and therefore, $f(t) = e^{ta}$. □

Take $t_1, t_2 \geq 0$. Set $X = t_1 A, Y = t_2 A$. Then X, Y commute and if we set $f(t) = e^{tX}e^{tY}$, then $f(0) = 1, f'(t) = Xe^{tX}e^{tY} + e^{tX}Ye^{tY} = (X+Y)e^{tX}e^{tY} = (X+Y)f(t)$. Hence by an appeal to the preceeding Lemma we have that $f(t) = e^{t(X+Y)}, \forall t \in \mathbb{R}$, so, in particular,$e^{(t_1+t_2)A} = e^{X+Y} = e^X e^Y = e^{t_1 A}e^{t_2 A}$. Thus, $S_{t_1+t_2} = S_{t_1}S_{t_2}$. Hence the proof.

Note for t sufficiently small ($t < 1$) we have that $\|S_t(x) - x\| \leq \|x\||t|(e^{\|A\|} - 1)$ which goes to 0 as $t \to 0$. □

6. Let $X = l_2$ and $\alpha = (\alpha_n) \in l_\infty$. Then show that the diagonal operator $A_\alpha : l_2 \to l_2$, defined as $A_\alpha x := (\alpha_n x_n)$ for $x = (x_n) \in l_2$ is compact iff $\alpha_n \to 0$.

Solution: Suppose that A_α is compact. Note $A_\alpha(e_n) = \alpha_n e_n$ so that each α_n is an eigenvalue of A_α where e_n denotes the sequence whose n-th term is 1 and all other terms are zero. Therefore $\alpha_n \to 0$. On the otherhand if $\alpha_n \to 0$, then if we define $T_n(x) = (\alpha_1 x_1, \alpha_2 x_2, \cdots, \alpha_n x_n, 0, 0, \cdots)$ then we see that each T_n is of finite rank (hence, compact) and note that $\|T_n(x) - A_\alpha(x)\|^2 = \sum_{m=n+1}^\infty |\alpha_m x_m|^2 \leq Sup_{m \geq n+1}|\alpha_m|^2\|x\|^2$ from which it immediately follows that $T_n \to A_\alpha$ and hence A_α is compact.

7. Let $H := L^2[0,1]$, (with Lebesgue measure). For $\phi \in L^\infty$, let $M_\phi : H \to H$. Show that $||M_\phi|| = ||\phi||_\infty$.

Solution: Note that for any $f \in H, \int |\phi f|^2 d\mu \leq \|\phi\|_\infty^2 \|f\|_2^2$ so that $M_\phi(f) \in H$ and it also follows that $\|M_\phi\| \leq \|\phi\|_\infty$. If possible let $\|M_\phi\| < \|\phi\|_\infty$. Then there is an $\epsilon > 0$ such that $\|M_\phi\| < \|\phi\|_\infty - \epsilon$. Thus there is a compact set K of $[0,1]$ (by regularity of μ) such that $\mu(K) > 0$ and $|\phi(x)| > \|M_\phi\| + \epsilon, \forall x \in K$. Observe that $\|M_\phi\|^2\mu(K) \geq \|M_\phi(\chi_K)\|^2 = \int |\phi\chi_K|^2 d\mu \geq (\|M_\phi\| + \epsilon)^2\mu(K)$ from which it follows that $\|M_\phi\| \geq \|M_\phi\| + \epsilon$, a contradiction. Thus, $\|M_\phi\| = \|\phi\|_\infty$.

$\square$

8. Let $H := L^2[0,1]$, (with Lebesgue measure), $A : H \to H, Af(x) := x^2 f(x)$. Then $A \geq 0, A \in B(H)$.

   a) Show that A has no cyclic vector in H. (Hint: Given $f \neq 0 \in H$, construct $g = g(f) \neq 0 \in H$, such that $< g, A^n f >_H = 0 \forall n \geq 1$.)

   b) Let H_e and H_o be the closed subspaces of H consisting respectively of the even and odd functions in H. Show that H_e and H_o are orthogonal subspaces of H and each is a cyclic subspace for A.

   Solution:

   (a) Given $f \neq 0$ in H. Define $g(x) = \overline{f(-x)} sgn(x), x \in [-1,1]$. Note $0 \neq g \in H$ and also note $A^n(f)(x) = x^{2n} f(x)$. Now

$$
\begin{aligned}
< A^n f, g > &= \int_{-1}^{1} x^{2n} f(x) f(-x) sgn(x) d\mu \\
&= -\int_{-1}^{0} x^{2n} f(x) f(-x) d\mu + \int_{0}^{1} x^{2n} f(x) f(-x) d\mu = 0
\end{aligned}
$$

   which shows that f is not a cyclic vector for A in H.

   (b) Given $f \in H_e, g \in H_o, < f, g >= \int_{-1}^{1} f\bar{g} d\mu = \int_{-1}^{0} f\bar{g} d\mu + \int_{0}^{1} f\bar{g} d\mu = -\int_{0}^{1} f\bar{g} d\mu + \int_{0}^{1} f\bar{g} d\mu = 0$. Thus H_e and H_o are orthogonal subspaces of H.
   Consider the function $f = 1 \in H_e$. Since $A^n(f)(x) = x^{2n}$, we see that $\{A^n f\}$ is the set of all monomials of even degree i.e equals the set $\{1, x^2, x^4, \cdots\}$, which is a total set in H_e so that H_e is A-cyclic subspace of H with f as a cyclic vector. Similarly one can see that the function $f(x) = x$ serves as a cyclic vector for H_o. Hence the proof.

$\square$